CHAPLYGIN'S EQUATIONS AND THE THEOREM OF THE ADDITIONAL MULTIPLIER IN THE CASE OF QUASICOORDINATES

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As is well known, the Chaplygin equations for the motion of a nonholonomic system [1] are derived on the assumption that the independent parameters are true coordinates. In [2], Chaplygin introduced a method for the integration of these equations, based on the use of a new variable τ, which is related to the time t by means of the differential relation

$$d\tau = N dt \qquad (0.1)$$

where N is a suitable function of the independent parameters. This function was called the additional multiplier by Chaplygin. However, already in the case of plane nonholonomic motion, which he uses to illustrate his theorem by means of an example, he in fact employs quasicoordinates (it must be noted that the general concept of quasicoordinates was introduced historically later), without taking due account of the situation. While Chaplygin's example is strictly correct, the extension of his theorem of the additional multiplier still remains without theoretical justification. Later, some authors (see, for example, [3]), expressed their doubts as to the validity of Chaplygin's theorem in the case of quasicoordinates. With the aid of superfluous coordinates, Shul'gin [4] showed that under certain conditions the form of Chaplygin's equations is preserved when some of the superfluous coordinates are quasicoordinates.

However, the theoretical justification of the equations of Chaplygin in the case of quasicoordinates has remained open. Let us mention the papers [6,7] of Novoselov, where an attempt is made to carry over Chaplygin's theorem to the case of nonlinear nonholonomic coordinates, but without an actual proof of the stated results. In the present paper Chaplygin's equations are extended to the case when the independent parameters are quasicoordinates; a theoretical justification of the method of the additional multiplier is given in this case; and there is given a class of problems in which the application of quasicoordinates lies within the limits of the theory of Chaplygin.

1. Chaplygin's equations for quasicoordinates. Consider a nonholonomic system, whose position is defined in terms of generalized coordinates $q_1, \ldots, q_n$. Suppose that there are m degrees of freedom $(m < n)$, i.e. there exist $n - m$ nonintegrable constraints relating the generalized velocities $\dot{q}_1, \ldots, \dot{q}_n$; these constraining relations are supposed to be linear and homogeneous. The coefficients in these equations, as well as the Lagrangian $\mathbf{L}$, depend only on the first m generalized coordinates.

Let us introduce as independent parameters the quasicoordinates $\pi_1, \ldots, \pi_m$ by means of m linear equations (here, and in what follows, repeated indices denote summation, as is customary):

$$\dot{\pi}_\alpha = a_{\alpha\beta}\dot{q}_\beta \qquad (\alpha, \beta = 1, \ldots, m) \tag{1.1}$$

where the coefficients $a_{\alpha\beta}$ are functions of $q_1, \ldots, q_m$. Using (1.1) and the $n - m$ equations of nonholonomic constraints, let us express all the generalized velocities $\dot{q}_1, \ldots, \dot{q}_m$ in terms of the m independent quasi-velocity $\dot{\pi}_1, \ldots, \dot{\pi}_m$:

$$\dot{q}_i = b_{i\sigma}\dot{\pi}_\sigma \qquad (i = 1, \ldots, n; \quad \sigma = 1, \ldots, m) \tag{1.2}$$

From this it follows that the variations of the generalized velocities are given in terms of the variations of the independent parameters as

$$\delta q_i = b_{i\alpha}\delta\pi_\alpha \qquad (i = 1, \ldots, n; \quad \alpha = 1, \ldots, m) \tag{1.3}$$

We shall begin with the d'Alembert-Lagrange equations written in generalized coordinates

$$\left(\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{q}_i} - \frac{\partial \mathbf{L}}{\partial q_i}\right)\delta q_i = 0 \qquad (i = 1, \ldots, n)$$

Substituting the δq_i from (1.3) into the last equation, we obtain a sum equal to zero; in view of the independence of the variations $\delta\pi_\alpha$, this sum leads to the m separate equations

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{q}_i}\right) b_{i\alpha} - \frac{\partial \mathbf{L}}{\partial q_i} b_{i\alpha} = 0 \qquad (i = 1, \ldots, n; \ \alpha = 1, \ldots, m) \tag{1.4}$$

In these equations, the Lagrangian $\mathbf{L}$ depends on the m coordinates $q_1, \ldots, q_m$ and the n velocities $\dot{q}_1, \ldots, \dot{q}_n$, i.e. $\mathbf{L} = \mathbf{L}(q, \dot{q}_i)$. Upon replacing the $\dot{q}_i$ by their values from (1.2), we obtain a function

$\mathbf{L}^* = \mathbf{L}^*(q_\beta, \pi_\sigma)$, for which

$$\mathbf{L}^*(q_\beta, \dot{\pi}_\sigma) = \mathbf{L}(q_\beta, b_{i\sigma}\dot{\pi}_\sigma) \tag{1.5}$$

From (1.5) it can be easily shown that

$$\frac{\partial \mathbf{L}^*}{\partial q_\beta} = \frac{\partial \mathbf{L}}{\partial q_\beta} + \frac{\partial \mathbf{L}}{\partial \dot{q}_i}\dot{\pi}_\sigma \frac{\partial b_{i\sigma}}{\partial q_\beta}, \qquad \frac{\partial \mathbf{L}}{\partial q_i} b_{i\alpha} = \frac{\partial \mathbf{L}^*}{\partial q_\beta} b_{\beta\alpha} - \frac{\partial \mathbf{L}}{\partial \dot{q}_i}\dot{\pi}_\sigma \frac{\partial b_{i\sigma}}{\partial q_\beta} b_{\beta\alpha}$$

$$\frac{\partial \mathbf{L}^*}{\partial \dot{\pi}_\sigma} = \frac{\partial \mathbf{L}}{\partial \dot{q}_i} b_{i\sigma}, \qquad \frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{q}_i}\right) b_{i\alpha} = \frac{d}{dt}\frac{\partial \mathbf{L}^*}{\partial \dot{\pi}_\alpha} - \frac{\partial \mathbf{L}}{\partial q_i}\frac{\partial b_{i\alpha}}{\partial \dot{q}_\beta} b_{\beta\sigma}\dot{\pi}_\sigma$$

$$(i = 1, \ldots, n; \quad \alpha, \beta, \sigma = 1, \ldots, m)$$

which, upon substitution into (1.4), making use of the notations

$$\frac{\partial b_{i\alpha}}{\partial q_\beta} b_{\beta\sigma} = \frac{\partial b_{i\alpha}}{\partial \pi_\sigma}, \qquad \frac{\partial \mathbf{L}^*}{\partial q_\beta} b_{\beta\alpha} = \frac{\partial \mathbf{L}^*}{\partial \pi_\alpha} \tag{1.6}$$

lead us to

$$\frac{d}{dt}\frac{\partial \mathbf{L}^*}{\partial \dot{\pi}_\alpha} - \frac{\partial \mathbf{L}^*}{\partial \pi_\alpha} + \frac{\partial \mathbf{L}}{\partial \dot{q}_i}\left(\frac{\partial b_{i\sigma}}{\partial \pi_\alpha} - \frac{\partial b_{i\alpha}}{\partial \pi_\sigma}\right)\dot{\pi}_\sigma = 0 \qquad \left(\begin{matrix} i = 1, 2, 3, \ldots, n \\ \alpha, \beta, \sigma = 1, \ldots, m \end{matrix}\right) \tag{1.7}$$

Equations (1.7) are Chaplygin's equations in quasicoordinates. It is readily seen that Equations (1.7) coincide with Chaplygin's equations [1] when $\pi_1, \ldots, \pi_m$ coincide with the true coordinates, i.e. when $\pi_\alpha = q_\alpha$ $(\alpha = 1, \ldots, m)$. Indeed, in this case

$$b_{\alpha\beta} = \delta_{\alpha\beta}, \qquad b_{\beta\sigma} = \delta_{\beta\sigma} \qquad (\delta_{\alpha\beta},\ \delta_{\beta\sigma} - \text{Kronecker's symbols})$$

and Equations (1.7) become

$$\frac{d}{dt}\frac{\partial \mathbf{L}^*}{\partial \dot{q}_\alpha} - \frac{\partial \mathbf{L}^*}{\partial q_\alpha} + \frac{\partial \mathbf{L}}{\partial \dot{q}_j}\left(\frac{\partial b_{j\sigma}}{\partial q_\alpha} - \frac{\partial b_{j\alpha}}{\partial q_\sigma}\right)\dot{q}_\sigma = 0 \qquad \left(\begin{matrix} \alpha, \sigma = 1, \ldots, m \\ j = m+1, \ldots, n \end{matrix}\right) \tag{1.8}$$

Equations (1.7) differ from the equations of Boltzmann-Hamel [5], in quasicoordinates, in the nonholonomic terms. In the construction of the coefficients γ^k_{ij} in the Boltzmann-Hamel equations one employs both the direct and the inverse matrices of the transformation linking the coordinates and the quasicoordinates. In the construction of the analogous terms in Equations (1.7) one employs coefficients from a single rectangular matrix (with n rows and m columns), which does not have an inverse.

2. Theorem of the additional multiplier in quasicoordinates. Consider a nonholonomic system of Chaplygin type with two degrees of freedom. As free parameters let us choose quasicoordinates π_1 and π_2, in terms of which all the true coordinates are given by

$$\dot{q}_i = b_{i1}\dot{\pi}_1 + b_{i2}\dot{\pi}_2 \qquad (i = 1, \ldots, n) \tag{2.1}$$

Suppose that the Lagrangian $\mathbf{L}$ depends only on q_1, q_2 and all the generalized velocities $\dot{q}_1, \ldots, \dot{q}_n$, and that the coefficients b_{i1} and b_{i2} are functions of q_1 and q_2. The equations of motion (1.7) in the system under consideration may be written as follows:

$$\frac{d}{dt}\frac{\partial \mathbf{L}^*}{\partial \dot{\pi}_1} - \frac{\partial \mathbf{L}^*}{\partial \pi_1} = \dot{\pi}_2 S, \qquad \frac{d}{dt}\frac{\partial \mathbf{L}^*}{\partial \dot{\pi}_2} - \frac{\partial \mathbf{L}^*}{\partial \pi_2} = -\dot{\pi}_1 S \tag{2.2}$$

where

$$S = \frac{\partial \mathbf{L}}{\partial \dot{q}_i}\left(\frac{\partial b_{i2}}{\partial \pi_1} - \frac{\partial b_{i1}}{\partial \pi_2}\right) \qquad (i = 1, \ldots, n) \tag{2.3}$$

and the function $\mathbf{L}^* = \mathbf{L}^*(q_1, q_2, \dot{\pi}_1, \dot{\pi}_2)$ is obtained from $\mathbf{L}$ by replacing the generalized velocities $\dot{q}_i$ by their equivalents from (2.1).

In order to transform Equations (2.2) let us introduce a new independent variable τ by means of Equation (0.1). Suppose that the kinetic energy T^* of the system is a quadratic form in the quasicoordinates $\dot{\pi}_1$ and $\dot{\pi}_2$. Using primes to denote differentiation with respect to τ, we obtain

$$2T^* = L_1\dot{\pi}_1^2 + 2M\dot{\pi}_1\dot{\pi}_2 + L_2\dot{\pi}_2 = N^2(L_1\pi_1'^2 + 2M\pi_1'\pi_2' + L_2\pi_2'^2) = 2T^\circ$$

$$\dot{\pi}_\beta = N\pi_\beta'$$

$$\frac{\partial T^*}{\partial \dot{\pi}_\beta} = \frac{\partial T^\circ}{\partial \pi_\beta'}\frac{\partial \pi_\beta}{\partial \dot{\pi}_\beta} = \frac{1}{N}\frac{\partial T^\circ}{\partial \pi_\beta'} \qquad \frac{\partial T^*}{\partial q_\alpha} = \frac{\partial T^\circ}{\partial q_\alpha} - \frac{1}{N}\frac{\partial N}{\partial q_\alpha}\frac{\partial T^\circ}{\partial \pi_\sigma'}\pi_\sigma' \qquad (\alpha, \beta, \sigma = 1, 2)$$

From these equations, in view of (1.6), we deduce that

$$\frac{\partial T^*}{\partial \pi_\beta} = \frac{\partial T^*}{\partial q_\alpha}b_{\alpha\beta} = \frac{\partial T^\circ}{\partial \pi_\beta} - \frac{1}{N}\frac{\partial N}{\partial q_\alpha}b_{\alpha\beta}\frac{\partial T^\circ}{\partial \pi_\sigma'}\pi_\sigma'$$

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\beta} = \frac{d}{d\tau}\frac{\partial T^\circ}{\partial \pi_\beta'} - \frac{1}{N}\frac{\partial N}{\partial q_\alpha}\frac{\partial T^\circ}{\partial \pi_\beta'}b_{\alpha\sigma}\pi_\sigma' \qquad (\alpha, \beta, \sigma = 1, 2)$$

Using these equations, Equations (2.2) become

$$\frac{d}{d\tau}\frac{\partial \mathbf{L}^\circ}{\partial \pi_1'} - \frac{\partial \mathbf{L}^\circ}{\partial \pi_1} = \pi_2' R, \qquad \frac{d}{d\tau}\frac{\partial \mathbf{L}^\circ}{\partial \pi_2'} - \frac{\partial \mathbf{L}^\circ}{\partial \pi_2} = -\pi_1' R \tag{2.4}$$

where $\mathbf{L}^\circ = T^\circ - V$, and the function R is defined by

$$R = NS - \frac{1}{N}\frac{\partial N}{\partial \pi_1}\frac{\partial T^\circ}{\partial \pi_2'} + \frac{1}{N}\frac{\partial N}{\partial \pi_2}\frac{\partial T^\circ}{\partial \pi_1'} \tag{2.5}$$

with

$$\frac{\partial N}{\partial \pi_\alpha} = \frac{\partial N}{\partial q_\beta}\frac{\partial q_\beta}{\partial \pi_\alpha} = \frac{\partial N}{\partial q_\beta} b_{\beta\alpha} \qquad (\alpha, \beta=1,2)$$

Equation (2.4) takes the ordinary form of the second-order Lagrange equations, provided that the function N is chosen so that the following equation holds:

$$R = 0 \tag{2.6}$$

Let us introduce canonical variables

$$p_1 = \frac{\partial T^\circ}{\partial \pi_1'} = N^2 (L_1\pi_1' + M\pi_2'), \qquad p_2 = \frac{\partial T^\circ}{\partial \pi_2'} = N^2 (M\pi_1' + L_2\pi_2') \tag{2.7}$$

Replacing in $\partial \mathrm{L}/\partial \dot{q}_i$ the quantities $\dot{q}_i$ by p_1 and p_2, by means of Equations (2.1) and (2.7), we obtain

$$\frac{\partial \mathrm{L}}{\partial \dot{q}_i} = \frac{1}{N}(A_i p_1 + A_i p_2) \tag{2.8}$$

where A_i, B_i are certain known functions of the generalized coordinates q_1 and q_2. Substituting now from (2.8) into (2.5), we arrive at an expression which is linear in p_1 and p_2. The requirement (2.6) will be automatically satisfied if the function N is such that

$$\frac{1}{N}\frac{\partial N}{\partial \pi_1} = B_i\left(\frac{\partial b_{i2}}{\partial \pi_1} - \frac{\partial b_{i1}}{\partial \pi_2}\right), \qquad \frac{1}{N}\frac{\partial N}{\partial \pi_2} = -A_i\left(\frac{\partial b_{i2}}{\partial \pi_1} - \frac{\partial b_{i1}}{\partial \pi_2}\right) \tag{2.9}$$

where

$$\frac{\partial b_{i\alpha}}{\partial \pi_\beta} = \frac{\partial b_{i\alpha}}{\partial q_\sigma}\frac{\partial q_\sigma}{\partial \pi_\beta} = \frac{\partial b_{i\alpha}}{\partial q_\sigma} b_{\sigma\beta} \qquad \begin{pmatrix} i = 1, \ldots, n \\ \alpha, \beta, \sigma = 1, 2\end{pmatrix}$$

It is readily seen that Equations (2.9) obtained here coincide with Equations (9) of Chaplygin [2] when the quasicoordinates π_1 and π_2 are the true coordinates. Indeed, setting $\pi_1 = q_1$, $\pi_2 = q_2$, it follows that $b_{11} = b_{22} = 1$, $b_{12} = b_{21} = 0$ and Equations (2.9) become

$$\frac{1}{N}\frac{\partial N}{\partial q_1} = B_j\left(\frac{\partial b_{j2}}{\partial q_1} - \frac{\partial b_{j1}}{\partial q_2}\right), \qquad \frac{1}{N}\frac{\partial N}{\partial q_2} = -A_j\left(\frac{\partial b_{j2}}{\partial q_1} - \frac{\partial b_{j1}}{\partial q_2}\right) \tag{2.10}$$

$$(j = 3, 4, \ldots, n)$$

which were given by Chaplygin.

3. A class of problems in which the introduction of quasicoordinates does not go beyond the limits of Chaplygin's theory. A comparison of Equations (1.7) and (1.8), as well as of (2.9) and (2.10), reveals that Chaplygin's equations and the equations for the additional multiplier may be written in the same form, both in the case of the true coordinates and in the case of quasicoordinates. Consequently, the extension of the theorem of the additional multiplier to the case of quasicoordinates has now been entirely justified. However, in the case of quasicoordinates, as contrasted with the case of true coordinates, in the computation of the partial derivatives one must use Equation (1.6). Hence, the final form of the equations in the case of quasicoordinates may differ from the final form of the equations in the case of true coordinates.

In spite of this, among Chaplygin systems there is a class of problems for which the final equations in both cases are identical. This class obeys the following two conditions:

1. The number l of true coordinates, on which depend the coefficients of the nonholonomic constraints and the Lagrangian, is smaller than the number m of degrees of freedom of the system.

2. The number k of quasicoordinates, which, together with l true coordinates, are chosen as independent parameters of the system, does not exceed the number $m - l$.

Let us show that, if these two conditions are valid, then Equations (1.7) and (2.9) coincide with the Chaplygin equations (1.8) and (2.10), respectively.

Indeed, suppose that in (1.2) the first l $(l < m)$ quasicoordinates are true coordinates, and that the coefficients $b_{i\sigma}$ depend only on $q_1, \ldots, q_l$. From this it follows that $b_{r\sigma} = \delta_{r\sigma}$ for $r, \sigma = 1, \ldots, l$ (where $\delta_{r\sigma}$ is Kronecker's symbol) and $b_{r\sigma} \equiv 0$ for $\sigma = l + 1,\ l + 2, \ldots, m$.

Suppose that the Lagrangian, in addition to generalized velocities, depends only on the coordinates $q_1, \ldots, q_l$. In the first l equations (1.7), in view of (1.6), we have

$$\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\left(\frac{\partial b_{i\sigma}}{\partial \pi_\alpha}-\frac{\partial b_{i\alpha}}{\partial \pi_\sigma}\right)\dot{\pi}_\sigma=\frac{\partial \mathrm{L}}{\partial \dot{q}_j}\left(\frac{\partial b_{js}}{\partial q_r}-\frac{\partial b_{jr}}{\partial q_s}\right)\dot{q}_s \qquad \begin{pmatrix} j=l+1,\ l+2,\ldots,n \\ r,\ s=1,2,\ldots,l \end{pmatrix}$$

and thus the l equations (1.7) become

$$\frac{d}{dt}\frac{\partial \mathrm{L}^*}{\partial \dot{q}_r}-\frac{\partial \mathrm{L}^*}{\partial q_r}+\frac{\partial \mathrm{L}}{\partial \dot{q}_j}\left(\frac{\partial b_{js}}{\partial q_r}-\frac{\partial b_{jr}}{\partial q_s}\right)\dot{q}_s=0 \qquad \begin{pmatrix} j=l+1,\ l+2,\ldots,n \\ r,\ s=1,2,\ldots,l \end{pmatrix} \quad (3.1)$$

Starting from Equations (1.8), let us remark that, in the class of problems under consideration, the index j in Equations (1.8) must take the values $l + 1,\ l + 2, \ldots, n$, since the $n - l$ generalized velocities $\dot{q}_{l+1}, \ldots, \dot{q}_n$ are expressible in terms of the independent parameters. Since the quantities $q_{l+1}, q_{l+2}, \ldots, q_n$ never appear, the corresponding derivatives also do not appear, and thus the first l equations of (1.8) coincide with (3.1).

It remains to verify that the remaining $m - l$ equations contain identical expressions, regardless of whether (1.7) or (1.8) is used as a starting point. In agreement with (1.6), we obtain

$$\frac{\partial \mathbf{L}^*}{\partial \pi_\rho} = \frac{\partial \mathbf{L}^*}{\partial q_r} b_{r\rho} = 0, \qquad \frac{\partial b_{i\sigma}}{\partial \pi_\rho} = \frac{\partial b_{i\sigma}}{\partial q_r} b_{r\sigma} = 0, \qquad \frac{\partial b_{i\rho}}{\partial \pi_\sigma} \dot{\pi}_\sigma = \frac{\partial b_{i\rho}}{\partial q_r} b_{r\sigma}\dot{\pi}_\sigma = \frac{\partial b_{j\rho}}{\partial q_r} \dot{q}_r$$

$$(i = 1, \ldots, n;\ j = l+1,\ l+2, \ldots, n;\ \rho = l+1,\ l+2, \ldots, m;\ r = 1, \ldots, l)$$

which, upon substitution into (1.7), yields

$$\frac{d}{dt}\frac{\partial \mathbf{L}^*}{\partial \dot{\pi}_\rho} - \frac{\partial \mathbf{L}}{\partial \dot{q}_j}\frac{\partial b_{j\rho}}{\partial q_r}\dot{q}_r = 0 \qquad \left(\begin{matrix} j = l+1,\ l+2, \ldots, n \\ \rho = l+1,\ l+2, \ldots, m \\ r = 1, 2, \ldots, l \end{matrix}\right) \tag{3.2}$$

Starting now from (1.8), the equations with indices $l + 1,\ l + 2, \ldots, m$ again lead to (3.2), because the quantities $\dot{q}_\rho$ $(\rho = l + 1,\ l + 2,\ \ldots,\ m)$ by their very definition coincide with $\dot{\pi}_\rho$.

Turning now to the equations of the additional multiplier, we must verify analogously the identity of the finite equations which are obtained from (2.9) and (2.10). For $m = 2$ one may add, without leaving the class of Chaplygin systems, only one quasicoordinate π_2 to a true coordinate q_1. Then we have, in (2.1), that

$$b_{11} = 1, \quad b_{12} = 0, \quad \pi_1 = q_1$$

$$\frac{\partial b_{i2}}{\partial \pi_1} = \frac{\partial b_{i2}}{\partial q_1} = \frac{\partial b_{j2}}{\partial q_1}, \qquad \frac{\partial b_{i1}}{\partial \pi_2} = \frac{\partial b_{i1}}{\partial q_1} b_{12} = 0 \qquad \left(\begin{matrix} i = 1, 2, \ldots, n \\ j = 2, 3, \ldots, n \end{matrix}\right)$$

and (2.9) takes the form

$$\frac{1}{N}\frac{\partial N}{\partial q_1} = B_j \frac{\partial b_{j2}}{\partial q_1}, \qquad \frac{1}{N}\frac{\partial N}{\partial \pi_2} = -A_j \frac{\partial b_{j2}}{\partial q_1} \qquad (j = 2, 3, \ldots, n) \tag{3.3}$$

Starting now from Equations (2.10), one must keep in mind that the index j takes the values $j = 2,\ 3,\ \ldots,\ n$, and that the coordinate q_2 does not appear explicitly in the coefficients b_{j2}. From this, we obtain immediately Equation (3.3) for the additional multiplier $N = N(q_2,\ \pi_2)$. Thus, we have obtained a class of problems in which both procedures coincide (to this class belong the cases considered in [4]). In this class we have the plane-parallel nonholonomic motions of Chaplygin [2], where Equations (2.1) have the form

$$\dot{\varphi} = \dot{\varphi}, \qquad \dot{x} = \dot{\pi}\cos\varphi, \qquad \dot{y} = \dot{\pi}\sin\varphi$$

with the angle ϕ and the coordinates x, y being true coordinates, while the arc length π is a quasicoordinate. Let us obtain the equations of the additional multiplier N, starting with Equations (2.9) in quasicoordinates. The kinetic energy T is

$$\frac{2T}{m} = [\dot{x} - \dot{\varphi}(\alpha\sin\varphi + \beta\cos\varphi)]^2 + [\dot{y} + \dot{\varphi}(\alpha\cos\varphi - \beta\sin\varphi)]^2 + k^2\dot{\varphi}^2$$

and Equations (2.8) become

$$A_1 = 1, \qquad A_2 = -\frac{\alpha}{\alpha^2 + k^2}\sin\varphi, \qquad A_3 = \frac{\alpha}{\alpha^2 + k^2}\cos\varphi.$$

$$B_1 = 0, \qquad B_2 = \cos\varphi - \frac{\alpha\beta}{\alpha^2 + k^2}\sin\varphi, \qquad B_3 = \frac{\alpha\beta}{\alpha^2 + k^2}\cos\varphi + \sin\varphi$$

while (2.9) is just

$$\frac{1}{N}\frac{\partial N}{\partial\varphi} = \frac{\alpha\beta}{\alpha^2 + k^2}, \qquad \frac{1}{N}\frac{\partial N}{\partial\pi} = -\frac{\alpha}{\alpha^2 + k^2}$$

which coincide with the equations given by Chaplygin.

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